INTERSECTION NUMBERS OF GEODESIC CURVES IN A SURFACE

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Abstract. For a compact surface X with negative curvature we show that the tails of the distribution $i(\alpha,\beta)/l(\alpha)l(\beta)$ are bounded by a decreasing exponential function (here, α and β are closed geodesics of X, $i(\alpha,\beta)$ denotes their intersection number and $l(\cdot)$ is the hyperbolic length function of X.) As a consequence we find the normalized average of the intersection numbers of pairs of closed geodesics. In addition, we prove that the size of the set of geodesics of length T whose self-intersection number is not close to $T^2/(2\pi^2(g-1))$ decrease also exponentially fast as $T\to\infty$, where g is the genus of X. As a corollary, we obtain a result of S. Lalley which states that most closed geodesics of length T have roughly $T^2/(2\pi^2(g-1))$ self-intersections, for T large.

1. Introduction

. Let X be a compact hyperbolic surface, $\mathcal{G}$ be the set of closed geodesics of X, $\mathcal{G}_t$ be the subset of $\mathcal{G}$ consisting of the geodesics whose length is at most t and $N(t)$ be the number of elements of $\mathcal{G}_t$. It is a classical result of G. Margulis (see [11, §6, Theorem 5]) that the number $N(t)$ satisfies the asymptotic formula $N(t)\sim e^t/t$, i.e., the ratio of the two sides converges to one, as $t\to\infty$.

. For a pair of geodesics α and β of X we denote by $i(\alpha,\beta)$ the (*geometric*) *intersection number of* α *and* β, that is, the number of points of intersection of α and β. In particular, $i(\gamma):=i(\gamma,\gamma)$ is the number of self-intersections of the geodesic γ.

. These numbers have been of interest to many researchers and here are some of the most relevant results so far achieved. S. Lalley showed in [10, Theorem 1] that for T large enough, the number of self-intersections of most of the closed geodesics of length T is $T^2/(2\pi^2(g-1))$, where g is the genus of X. Later, M. Pollicot and R. Sharp generalized this result to self-intersections of closed geodesics with an angle in a given interval (see [13, Theorem 1].) And recently, M. Chas and S. Lalley in [8] proved that if a class of a geodesic is chosen at random from among all classes of m letters, then the distribution of the self-intersection numbers approaches the Gaussian distribution, for m "large enough". Furthermore, M. Chas and S. Lalley also proved in [9] that for a certain constant $\kappa>0$ the random variable $(N_T-\kappa T^2)/T$ has a limit distribution as $T\to\infty$, where N_T is the number of self-intersections of a closed geodesic of X of length $\leq T$ randomly chosen.

. In this paper we prove that the tails of the distribution $i(\alpha,\beta)/l(\alpha)l(\beta)$ are bounded by a decreasing exponential function.

Theorem 1. *Let $\epsilon>0$. There exists $\eta>0$ such that*

$$\frac{1}{N(s)N(t)}\#\left\{(\alpha,\beta)\in\mathcal{G}_s\times\mathcal{G}_t:\left|\frac{i(\alpha,\beta)}{l(\alpha)l(\beta)}-\frac{1}{2\pi^2(g-1)}\right|>\epsilon\right\}=O(e^{-\eta\min\{s,t\}}),$$

as $s,t\to\infty$.

. Theorem 1 allows us to show that the normalized average of pairs of closed geodesics of lengths at most s and t is asymptotically equal to $1/(2\pi^2(g-1))$, as these lengths become arbitrarily large.

Theorem 2. $\displaystyle\frac{1}{N(s)N(t)}\sum_{(\alpha,\beta)\in\mathcal{G}_s\times\mathcal{G}_t}\frac{i(\alpha,\beta)}{l(\alpha)l(\beta)}\sim\frac{1}{2\pi^2(g-1)}$, *as* $s,t\to\infty$.

. In a similar way, we prove that the size (ν_L-measure) of the set of geodesics of length T whose self-intersection number is not close to $T^2/(2\pi^2(g-1))$ decrease also exponentially fast as $T\to\infty$. For a geodesic (not necessarily closed) γ, let denote by γ^T the restriction of γ to its first segment of length T.

Theorem 3. *For every* $\epsilon>0$ *there exists* $\eta>0$ *such that*

$$\nu_L(\{\mathbf{v}\in T_1(X):|i(\gamma_{\mathbf{v}}^T)/T^2-1/(2\pi^2(g-1))|\geq\epsilon\})=O(e^{-\eta T}).$$

. As a consequence we obtain the following result that was proven by Lalley in [10, Theorem 1].

Corollary 1 (Lalley)**.** *For every* $\epsilon>0$

$$\lim_{T\to\infty}\frac{1}{N(T)}\#\{\gamma\in\mathcal{G}_T:|i(\gamma)/T^2-1/(2\pi^2(g-1))|<\epsilon\}=1.$$

. The outline of this paper is the following. Section 2 is the collection of definitions and results needed in the demonstrations of Theorems 1 and 2, and Section 3 contains the proofs of these theorems as well as the proofs of Theorem 3 and Corollary 1. For detailed explanation of all the concepts (or different approach) used in this article, please see [3], [4], [5] and [13].

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2. Preliminaries

2.1. **Tangent Bundles.** Let $T_1(X)=\{(x,v)\mid x\in X,v\in T_x(X),\|v\|=1\}$ be the *unit tangent bundle* of X, $\Phi=\{\phi^t\}$ be the *geodesic flow* of $T_1(X)$ and $\mathcal{F}$ be the *foliation* of $T_1(X)$ by Φ-*orbits*. Note that the Φ-orbits are the geodesics of X. For $\mathbf{v}=(x,v)\in T_1(X)$, let $\gamma_{\mathbf{v}}$ denote the geodesic such that $\gamma_{\mathbf{v}}(0)=x$ and $\dot\gamma_{\mathbf{v}}(0)=v$.
. Consider $\mathscr{E}=\{(x,v,w):(x,v),(x,w)\in T_1(X),u\neq v\}$. Let $p_1,p_2:\mathscr{E}\to T_1(X)$ be defined by $p_1((x,v,w))=(x,v)$ and $p_2((x,v,w))=(x,w)$, respectively.
. Denote by $\mathscr{P}=\mathscr{P}(T_1(X))$ the set of the Φ-invariant probability measures in $T_1(X)$ equipped with the weak*-topology. Let $h(\mu)$ denote the *measure theoretic entropy of* Φ with respect to $\mu\in\mathscr{P}$ and $h:=\max_{\mu\in\mathscr{P}}h(\mu)$ (please see [5, §4.3] for definitions). For a compact hyperbolic surface there is a unique Φ-invariant probability measure in $T_1(X)$ with maximum entropy $h=1$, which is the normalized Riemannian measure also called the (*normalized*) *Liouville* measure, denoted by ν_L. In our case, this measure coincides with μ_{BM}, the Bowen-Margulis probability measure. (see [12, Proposition 10].)
. We will use the characterization of ν_L given in [5, Theorem 20.1.3]. For $\gamma\in\mathcal{G}$, let $\zeta_\gamma:=\int_0^{l(\gamma)}\delta_{\phi^s\mathbf{v}}\mathrm{d}s$, with $\mathbf{v}\in\gamma$ and δ_y denoting the probability measure with support $\{y\}$.

Theorem 4.

$$\nu_L = \mu_{BM} = \lim_{t\to\infty} \frac{1}{N(t)} \sum_{\gamma\in\mathcal{G}_t} \frac{\zeta_\gamma}{l(\gamma)}.$$

2.2. **Current Measures and Intersection Form.**
. For each Φ-invariant finite measure μ of $T_1(X)$ (not necessarily a probability measure) there exists an associated transverse measure to $\mathcal{F}$, which we denote by $\widetilde{\mu}$. The set of all of these transverse measures equipped with the weak*-topology is known as the *space of current measures* of X and we denote it by $\mathcal{C}$. Each $\widetilde{\mu}\in\mathcal{C}$ is normalized by the requirement that (locally) $\mu = \widetilde{\mu}\times \mathrm{d}t$, where $\mathrm{d}t$ is the one-dimensional Lebesgue measure along orbits in $\mathcal{F}$.
. Consider the foliations $\mathcal{F}_1 = p_1^{-1}(\mathcal{F})$ and $\mathcal{F}_2 = p_2^{-1}(\mathcal{F})$ of $T_1(X)$, for p_1 and p_2 as defined in Section 2.1. Furthermore, for $\widetilde{\mu},\widetilde{\nu}\in\mathcal{C}$, define $\widehat{\mu}_1 = p_1^{-1}(\widetilde{\mu})$ and $\widehat{\nu}_2 = p_2^{-1}(\widetilde{\nu})$. Note that the new measures $\widehat{\mu}_1$ and $\widehat{\nu}_2$ are transverse to $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. The *intersection form of* $\widetilde{\mu}$ *and* $\widetilde{\nu}$, denoted by $\imath(\widetilde{\mu},\widetilde{\nu})$, is defined as the total mass of $\mathscr{E}$ with respect to the product measure $\widehat{\mu}_1\times\widehat{\nu}_2$, that is, $\imath(\widetilde{\mu},\widetilde{\nu}) = \int_{\mathscr{E}} \mathrm{d}(\widehat{\mu}_1\times\widehat{\nu}_2) = (\widehat{\mu}_1\times\widehat{\nu}_2)(\mathscr{E})$. In addition, for $\widetilde{\mu}\in\mathcal{C}$ define $\ell(\widetilde{\mu})$, by $\ell(\widetilde{\mu}) = \imath(\widetilde{\mu},\widetilde{\nu_L})$.
. Now, for every $\gamma\in\mathcal{G}$ there exists a unique Φ-invariant measure μ^γ of total mass $l(\gamma)$, where $l(\gamma)$ is the length of the geodesic γ with respect to the hyperbolic metric of X. This measure μ^γ is supported on the orbit of γ. Let $\widetilde{\mu^\gamma}$ denote the corresponding transverse measure to the orbit foliation $\mathcal{F}$. These type of current measures are simply finite sums of Dirac measures on the quotient of pairs of closed geodesics on $\widetilde{X}$ (the universal cover of X) consisting of lifts of γ and its $\pi_1(X)$-images (see [2, §1]).
. By identifying the current measure $\widetilde{\mu^\gamma}$ with the corresponding closed geodesic γ F. Bonahon showed the following properties of the Liouville current $\widetilde{\nu}_L$ as well as the properties of the functions $\imath$ and ℓ which we will use later on this paper (see [3, Propositions 14 and 15] and [2, Proposition 4.5].)

Theorem 5 (Bonahon)**.** *Consider the functions $\imath$ and ℓ as defined above. Then*

(1) *The function $\imath$ is a continuous extension of the intersection number function on $\mathcal{G}\times\mathcal{G}$. In particular, $\imath(\widetilde{\mu^\alpha},\widetilde{\mu^\beta}) = i(\alpha,\beta)$. The Liouville current satisfies $\imath(\widetilde{\nu_L}) = \pi^2|\chi(X)| = 2\pi^2(g-1)$.*
(2) *The function ℓ is a continuous extension of the length function on $\mathcal{G}$. In particular, if γ is a closed geodesic of X then $\ell(\widetilde{\mu^\gamma}) = l(\gamma)$ where $l(\gamma)$ is the length of γ. The Liouville current satisfies $\ell(\widetilde{\nu_L}) = \pi^2|\chi(X)| = 2\pi^2(g-1)$.*

3. Proofs of the main results

. The proofs of Theorems 1 and 3 are basically based on the large deviation results proven by Yuri Kifer in [6, Theorem 3.4] and [7, Theorem 2.1].
. For $t>0$ and $\mathbf{v}\in T_1(X)$ consider the following measure $\zeta_{\mathbf{v}}^t := \int_0^t \delta_{\phi^s\mathbf{v}}\mathrm{d}s$. In particular, $\zeta_{\mathbf{v}}^{l(\gamma_{\mathbf{v}})} = \zeta_{\gamma_{\mathbf{v}}}$, for $\gamma_{\mathbf{v}}\in\mathcal{G}$.

Theorem 6 (Kifer)**.** *For any closed $K\subset\mathscr{P}$,*

$$\lim_{T\to\infty}\frac{1}{T}\log\nu_L(\{\mathbf{v}\in T_1(X) : \zeta_{\mathbf{v}}^T/T\in K\}) \le -\inf\{1-h(\nu):\nu\in K\}.$$

. For $\mathbf{v} \in T_1(X)$, let $\gamma_{\mathbf{v}}$ be the geodesic such that $\mathbf{v} \in \gamma_{\mathbf{v}}$. And let $\mathbf{v}_\gamma = (x, v) \in T_1(X)$ be a vector of the geodesic γ, i.e., $\gamma(t) = x$ and $\dot{\gamma}(t) = v$, for some $t \in \mathbb{R}$.

Proof of Theorem 3. Consider the continuous function

$$\begin{aligned} \mathfrak{f}: \quad \mathscr{P} &\to \mathbb{R} \\ \nu &\mapsto \imath(\widetilde{\nu})/\ell(\widetilde{\nu})^2. \end{aligned}$$

Let $\epsilon > 0$ and take $K := \{\nu \in \mathscr{P} : |\mathfrak{f}(\nu) - \mathfrak{f}(\nu_L)| \geq \epsilon\}$. Note that K is a closed set because of the continuity of $\mathfrak{f}$, hence, K is compact since $\mathscr{P}$ is compact.
. By Theorem 6,

$$\lim_{T\to\infty} \frac{1}{T} \log \nu_L(\{\mathbf{v} \in T_1(X) : \zeta_{\mathbf{v}}^T/T \in K\}) \leq -\inf\{1 - h(\nu) : \nu \in K\}.$$

. Thus,

$$\begin{aligned} \nu_L(\{\mathbf{v} \in T_1(X) :& |i(\gamma_{\mathbf{v}}^T)/T^2 - 1/(2\pi^2(g-1))| \geq \epsilon\}) \\ &= \nu_L(\{\mathbf{v} \in T_1(X) : \zeta_{\mathbf{v}}^T/T \in K\}) = O(e^{-\eta T}), \end{aligned}$$

where $\eta := \inf\{1 - h(\nu) : \nu \in K\}$.
. Since ν_L is the unique measure with maximum entropy $h(\nu_L) = 1$ and $\nu_L \notin K$, we always have $\eta > 0$. $\square$

Proof of Corollary 1. Let $T, \epsilon > 0$. Consider the set

$$\mathscr{O}(T, \epsilon) := \{\gamma \in \mathcal{G}_t : |i(\gamma)/T^2 - 1/(2\pi^2(g-1))| \geq \epsilon\}.$$

Since $\mathscr{O}(T, \epsilon) = \mathcal{G}_T \setminus \{\gamma \in \mathcal{G}_T : |i(\gamma)/T^2 - 1/(2\pi^2(g-1))| < \epsilon\}$, it is enough to prove that

$$\lim_{T\to\infty} \frac{1}{N(T)} \#\mathscr{O}(T, \epsilon) = 0.$$

. Let $\varepsilon > 0$. By Theorem 5, there is $R_1 := R_1(\varepsilon)$ such that for every $T > R_1$,

$$\frac{1}{N(t)} \sum_{\gamma \in \mathcal{G}_t} \frac{\zeta_\gamma}{l(\gamma)} (\mathscr{O}(T, \epsilon)) < \nu_L(\mathscr{O}(T, \epsilon)) + \varepsilon/2.$$

. And, by Theorem 3, there exist $\eta := \eta(\epsilon), C := C(\epsilon), R_2 := R_2(\epsilon) > 0$ such that for $t \geq R_2$,

$$\nu_L(\mathscr{O}(t, \epsilon)) \leq Ce^{-\eta t} < \varepsilon/2.$$

. Thus, taking $R = \max\{R_1, R_2\}$, we get for $T > R$ that

$$\frac{1}{N(T)} \#\mathscr{O}(T, \epsilon) = \frac{1}{N(T)} \sum_{\gamma \in \mathcal{G}_T} \frac{\zeta_\gamma}{l(\gamma)} (\mathscr{O}(T, \epsilon)) \leq Ce^{-\eta T} + \varepsilon/2 < \varepsilon.$$

. Since ε was arbitrarily chosen, we conclude

$$\lim_{T\to\infty} \frac{1}{N(T)} \#\mathscr{O}(T, \epsilon) = 0$$

and the result of the corollary. $\square$

. Another deviation result by Y. Kifer given in [7, Theorem 2.1], similar to Theorem 6, states that the portion of "irregular" geodesics vanishes exponentially fast.

Theorem 7 (Kifer). *Let $\mathcal{U}$ be an open neighborhood of ν_L in $\mathscr{P}$. Then, there exists $\eta > 0$ such that*

$$\frac{1}{N(t)}\#\{\gamma \in \mathcal{G}_t : \mu^\gamma/l(\gamma) \notin \mathcal{U}\} = O(e^{-\eta t}),$$

as $t \to \infty$. Moreover, $\eta = \inf_{\nu\in\mathcal{U}^c}\{1 - h(\nu)\}$.

. Theorems 5 and 7 imply our main result Theorem 1.

Proof of Theorem 1. Let $s, t, \epsilon > 0$ with $s \leq t$ and consider the following function

$$\begin{aligned} \mathfrak{h} : \mathscr{P} \times \mathscr{P} &\to \mathbb{R} \\ (\widetilde{\mu}, \widetilde{\nu}) &\mapsto \imath(\widetilde{\mu}, \widetilde{\nu})/(\ell(\widetilde{\mu})\ell(\widetilde{\nu})). \end{aligned}$$

. The function $\mathfrak{h}$ is continuous because it is the quotient of two continuous functions, the intersection form function (by Theorem 5) and the function that assigns to a pair of measures the product of the lengths of their associated currents. As a result, the set $\mathcal{Z} = \mathfrak{h}^{-1}(1/(2\pi^2(g-1)) - \epsilon, 1/(2\pi^2(g-1)) + \epsilon)$, the preimage of the ball of radius ϵ centered at $\mathfrak{h}(\widetilde{\nu_L}, \widetilde{\nu_L}) = \imath(\widetilde{\nu_L})/\ell(\widetilde{\nu_L})^2 = 1/(2\pi^2(g-1))$, is an open subset of $\mathscr{P} \times \mathscr{P}$.

. Let $\mathcal{W}_{s,t} = \{(\alpha, \beta) \in \mathcal{G}_s \times \mathcal{G}_t : |(\widehat{\mu_1^\alpha} \times \widehat{\mu_2^\beta})(\mathscr{E})/l(\alpha)l(\beta) - 1/(2\pi^2(g-1))| < \epsilon\}$.
. Note that $\mathcal{W}_{s,t} = \{(\alpha, \beta) \in \mathcal{G}_s \times \mathcal{G}_t : (\mu^\alpha/l(\alpha), \mu^\beta/l(\beta)) \in \mathcal{Z}\}$.
. Since $\mathcal{Z}$ is an open set of the product topology of $\mathscr{P} \times \mathscr{P}$, there exist $\mathcal{U}, \mathcal{V} \subseteq \mathscr{P}$ open neighborhoods of ν_L such that $\mathcal{U} \times \mathcal{V} \subseteq \mathcal{Z}$.
. Consider the sets $\mathcal{U}_s := \{\gamma \in \mathcal{G}_s : \mu^\gamma/l(\gamma) \in \mathcal{U}\}$ and $\mathcal{V}_t := \{\gamma \in \mathcal{G}_t : \mu^\gamma/l(\gamma) \in \mathcal{V}\}$. Then, $\mathcal{U}_s \times \mathcal{V}_t \subseteq \mathcal{W}_{s,t}$.
. By Theorem 7, there exist $\eta_1 := \eta_1(\mathcal{U}), \eta_2 := \eta_2(\mathcal{V}), T_1, T_2, C_1, C_2 > 0$ such that for $s \geq T_1$ and $t \geq T_2$,

$$\frac{\#\mathcal{G}_s \setminus \mathcal{U}_s}{N(s)} \leq \frac{C_1}{e^{\eta_1 s}} \quad \text{and} \quad \frac{\#\mathcal{G}_t \setminus \mathcal{V}_t}{N(t)} \leq \frac{C_2}{e^{\eta_2 t}}.$$

. Thus, taking $T = \max\{T_1, T_2\}, C = C_1 + C_2 + C_1C_2$ and $\eta = \min\{\eta_1, \eta_2\}$, we get

$$\begin{aligned} \frac{\#(\mathcal{G}_s \times \mathcal{G}_t) \setminus \mathcal{W}_{s,t}}{N(s)N(t)} &\leq \frac{\#(\mathcal{G}_s \times \mathcal{G}_t) \setminus (\mathcal{U}_s \times \mathcal{V}_t)}{N(s)N(t)} \\ &= \frac{\#\mathcal{G}_s \setminus \mathcal{U}_s \cdot \#\mathcal{G}_t \setminus \mathcal{V}_t}{N(s)N(t)} + \frac{\#\mathcal{G}_s \setminus \mathcal{U}_s \cdot \#\mathcal{V}_t}{N(s)N(t)} + \frac{\#\mathcal{G}_t \setminus \mathcal{V}_t \cdot \#\mathcal{U}_s}{N(s)N(t)} \\ &\leq \frac{C_1C_2}{e^{\eta_1 s}e^{\eta_2 t}} + \frac{C_1}{e^{\eta_1 s}} + \frac{C_2}{e^{\eta_2 t}} \leq \frac{C}{e^{\eta t}}, \end{aligned}$$

whenever $T \leq s \leq t$. Hence, we conclude the result of the theorem. □

. The other key point for the proof of Theorem 2 is to find an adequate bound for the intersection numbers of the pairs of closed geodesics of X. This was achieved by Basmajian in [1, Thoerem 1.2]. Here we give a different bound with a different proof. This bound can also be deduced by Basmajian's techniques.

Proposition 1 (Basmajian). *Let $\alpha, \beta \in \mathcal{G}$ and $\varepsilon_0 = \operatorname{inj} X$, the injectivity radius of X. Then $i(\alpha, \beta) \leq 4l(\alpha)l(\beta)/\varepsilon_0^2$.*

Proof. Let $\alpha : [0, l(\alpha)] \to X$ and $\beta : [0, l(\beta)] \to X$ be closed geodesics of X. Let $\bar{\alpha}$ be a subarc of α of length less than $\varepsilon_0/2$ for which $i(\bar{\alpha}, \beta)$ is the largest. Hence, $i(\alpha, \beta) \leq ([l(\alpha)/(\varepsilon_0/2)] + 1) i(\bar{\alpha}, \beta)$. Let $\{x_1, x_2, \ldots, x_n\}$ be the ordered set of points of intersection of $\bar{\alpha}$ and β, where $\beta^{-1}(x_i) \leq \beta^{-1}(x_{i+1})$, for $1 \leq i \leq n-1$, and $n = i(\bar{\alpha}, \beta)$. Let β_k be the subarc of β from x_k to x_{k+1}, for $1 \leq k \leq n-1$ and β_n the subarc of β joining x_1 and x_n which does not intersect $\bar{\alpha}$. Similarly, let $\bar{\alpha}_k$ be the subarc of α from x_k to x_{k+1}, for $1 \leq k \leq n-1$, and $\bar{\alpha}_n$ be the subarc joining x_1 and x_n not intersecting $\bar{\alpha}$. Consider γ_k the concatenation of $\bar{\alpha}_k$ and β_k, for $1 \leq k \leq n$. Thus, γ_k is an essential loop of X, for $1 \leq k \leq n$. Hence, $\varepsilon_0 < l(\gamma_k) = l(\bar{\alpha}_k) + l(\beta_k) \leq \varepsilon_0/2 + l(\beta_k)$, which implies $\varepsilon_0/2 \leq l(\beta_k)$, for $1 \leq k \leq n$. . Consequently, $n < l(\beta)/(\varepsilon_0/2)$. Therefore,

$$i(\alpha, \beta) \leq \left(\left[\frac{l(\alpha)}{\varepsilon_0/2} \right] + 1 \right) i(\bar{\alpha}, \beta) \leq \frac{l(\alpha)}{\varepsilon_0/2} \frac{l(\beta)}{\varepsilon_0/2} = \frac{4 l(\alpha) l(\beta)}{\varepsilon_0^2}.$$

□

. Theorem 2 is a straightforward result of the following result, whose proof is a simple computation which uses Theorem 1.

Theorem 8. $\displaystyle \frac{1}{N(s)N(t)} \sum_{(\alpha,\beta) \in \mathcal{G}_s \times \mathcal{G}_t} \frac{(\widehat{\mu_1^\alpha} \times \widehat{\mu_2^\beta})(\mathscr{E})}{l(\alpha) l(\beta)} \sim \frac{1}{2\pi^2(g-1)}$ *as* $s, t \to \infty$.

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